

Latent Quaternionic Geometry

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Abstract

In this article we discuss the interaction between the geometry of a quaternion-Kähler manifold M and that of the Grassmannian $\mathbb{G}_3(\mathfrak{g})$ of oriented 3-dimensional subspaces of a compact Lie algebra \mathfrak{g} . This interplay is described mainly through the moment mapping induced by the action of a group G of quaternionic isometries on M . We give an alternative expression for the endomorphisms I_1, I_2, I_3 , both in terms of the holonomy representation of M and the structure of the Grassmannian's tangent space. A correspondence between the solutions of respective twistor-type equations on M and $\mathbb{G}_3(\mathfrak{g})$ is provided.

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1 Introduction

Let G be a compact Lie group acting by quaternionic isometries on a quaternion-Kähler (QK) manifold M . In this case a Killing vector field X satisfies the condition $L_X \Omega = 0$, where Ω is the parallel 4-form of the QK structure. Recall that the fibre of the standard rank 3 vector bundle over M (whose complexification is often written $S^2 H$) is isomorphic to $\mathfrak{sp}(1)$, and is spanned by a basis of endomorphisms I_1, I_2, I_3 satisfying the quaternionic relations

$$I_i^2 = -\text{Id} \quad \text{and} \quad I_i I_j = \epsilon^{ijk} I_k$$

with ϵ^{ijk} the sign of the permutation.

We denote by μ the moment map for the G action, and by μ_A the section of $S^2 H$ obtained by the contraction of μ with $A \in \mathfrak{g}$ through the metric induced by the Killing form. It satisfies the equation

$$d\mu_A = i(\tilde{A})\Omega, \tag{1}$$

where \tilde{A} is the Killing vector field generated by A (see [10], [11]). Another way of describing the sections coming from the moment map is expressed by the formula

$$\mu_A = \pi_{S^2 H}(\nabla \tilde{A})$$

up to a constant. The moment map μ is G -equivariant with respect to the given action of G on M and of the adjoint representation of G on \mathfrak{g} : it can be used to construct the G -equivariant morphism

$$\Psi : M_0 \longrightarrow \mathbb{G}_3(\mathfrak{g}),$$

where M_0 is an appropriate subset of M . The morphism Ψ was introduced by Swann ([22], [23]), who studied the unstable manifolds for the gradient flow of an appropriate functional ψ on this type of Grassmannians, proving that they admit a QK structure; we will use the map Ψ in order to relate in various ways the geometry of QK manifolds to that of Grassmannians of type $\mathbb{G}_3(\mathfrak{g})$.

In Section 2, we introduce the natural first-order differential operator D on the tautological rank k vector bundle over a Grassmannian $\mathbb{G}_k(\mathbb{R}^n)$, which annihilates projections of constant sections. Indeed, we show that all solutions of D arise in this way (Theorem 2.2). This illustrates a well-known technique, whereby solutions of an overdetermined differential operator may be interpreted as parallel sections of some connection on a larger bundle ([6]). Although quaternionic geometry and Lie algebras are not yet involved, we aim to show that D is completely analogous to the more complicated *twistor operator* \mathcal{D} on a QK manifold.

In Section 3, we recall the definition of \mathcal{D} on sections of the vector bundle S^2H , and explain that it is satisfied by the moment sections μ_A defined above. We then prove that under suitable hypotheses the map Ψ can be used to relate elements in $\ker \mathcal{D}$ with those in $\ker D$ where D now acts on the tautological rank 3 vector bundle V over $\mathbb{G}_3(\mathfrak{g})$.

Whilst the tangent space to $\mathbb{G}_3(\mathfrak{g})$ at V is given by

$$T_V \mathbb{G}_3(\mathfrak{g}) \cong V \otimes V^\perp, \quad (2)$$

the complexified tangent space to M has the form $H \otimes_{\mathbb{C}} E$, reflecting the representation of the holonomy group $Sp(1)Sp(n)$. Part of our problem is to reconcile the roles of the “auxiliary” vector bundles V and H with respective fibres \mathbb{R}^3 and \mathbb{C}^2 . In Section 4 we give an alternative description of the imaginary quaternion endomorphisms I_i over a point $x \in M$ in terms of $Sp(1)$ representations of a subgroup $Sp(1)$ diagonally embedded in $Sp(1)Sp(n)$.

In Section 5 we state our main results: we show that it is possible to push forward the endomorphisms I_1, I_2, I_3 so that they can be described as endomorphisms of the subspace $\Psi_* T_x M$ of (2), where $V = \Psi(x)$. In other words, if $Z = \sum_{i=1}^3 v_i \otimes p_i$ belongs to $\Psi_* T_x M$, then we can write

$$I_k Z = \sum_{i=1}^3 v_i \otimes q_i$$

and we shall explicitly determine the q_i s in terms of v_i, p_i and I_k . This is accomplished in Proposition 5.4, itself a geometric counterpart to the representation-theoretic Proposition 4.1.

Finally, in Section 6 we apply the theory to the case of an $Sp(1) \times Sp(1)$ action on \mathbb{HP}^1 and to other compatible examples. We describe some natural real 4-dimensional subspaces of (2) which correspond to quaternionic lines in $T_x M$, and are tangent to quaternion projective lines in Wolf spaces.

2 Operators on Grassmannians

Consider an n -dimensional real vector space \mathbb{R}^n equipped with an inner product $\langle \cdot, \cdot \rangle$; we can construct the Grassmannian of oriented k -planes $\mathbb{G}_k(\mathbb{R}^n)$, whose tangent space at a k -plane V can be identified with the linear space

$$\text{Hom}(V, V^\perp) \cong V^* \otimes V^\perp;$$

in fact if v_1, \dots, v_k is an orthonormal (ON) basis for V and w_1, \dots, w_{n-k} for V^\perp , then each homomorphism T_{ij} defined as $T_{ij}(v_k) = \delta_k^i w_j$, corresponds to an independent tangent direction; more explicitly, the curve

$$\alpha_{ij}(r) := \text{span}\{v_1, \dots, (\cos r)v_i + (\sin r)w_j, \dots, v_k\} \quad (3)$$

satisfies $\alpha_{ij}(0) = V$ and $\alpha'_{ij}(0) = T_{ij}$. The presence of a metric on V , induced from the ambient space \mathbb{R}^n , will allow us to write $V \otimes V^\perp$, using contraction via the metric for the isomorphism $V \cong V^*$.

We will be interested in studying differential operators and sections of vector bundles on $\mathbb{G}_k(\mathbb{R}^n)$, so we start by describing some induced objects. Given the metric, we have the splitting of the trivial bundle $\mathbb{G}_k(\mathbb{R}^n) \times \mathbb{R}^n$ in two subbundles: the tautological one \mathbf{V} and its orthogonal complement:

$$\begin{array}{ccc} \mathbf{V} \oplus \mathbf{V}^\perp & \xrightarrow{\cong} & \mathbb{G}_k(\mathbb{R}^n) \times \mathbb{R}^n \\ & \searrow p & \downarrow p' \\ & & \mathbb{G}_k(\mathbb{R}^n) \end{array} .$$

The presence of this metric also allows us to define connections on these two subbundles merely by composing d with the two projections π and π^\perp . For instance

$$\nabla^{\mathbf{V}} s = \pi ds,$$

where $s \in \Gamma(\mathbf{V})$ and d is the derivation in \mathbb{R}^n . To prove that this is a connection let a be a function, and note that

$$\begin{aligned} \nabla^{\mathbf{V}}(as) &= \pi d(as) = \pi((da)s + a(ds)) \\ &= (da)s + a\pi(ds) \\ &= (da)s + a\nabla^{\mathbf{V}} s \end{aligned}$$

as required. Moreover this connection is compatible with the metric induced on the fibres of \mathbf{V} by their ambient space \mathbb{R}^n : in fact if $s, t \in \Gamma(\mathbf{V})$ and $X \in T_V \mathbb{G}_k(\mathbb{R}^n)$ we have

$$\begin{aligned} X\langle s, t \rangle &= \langle Xs, t \rangle + \langle s, Xt \rangle = \langle \pi Xs, t \rangle + \langle s, \pi Xt \rangle \\ &= \langle \nabla_X^{\mathbf{V}} s, t \rangle + \langle s, \nabla_X^{\mathbf{V}} t \rangle. \end{aligned}$$

On the other hand we obtain the corresponding second fundamental form by projecting in the opposite way:

$$\Gamma(\mathbf{V}) \longrightarrow \Gamma(T^* \mathbb{G}_k(\mathbb{R}^n) \otimes \mathbf{V}^\perp)$$

which sends s to $\pi^\perp ds$; analogously II^\perp sends $s \in \Gamma(\mathbf{V}^\perp)$ to πds . Both II and II^\perp are tensors. In fact, if for example $s \in \Gamma(\mathbf{V}^\perp)$ and a is a function, we get

$$\pi d(as) = \pi(d(a)s + ad(s)) = \pi ad(s) = a\pi ds$$

so that we can think to II^\perp as a section of the bundle

$$\text{Hom}(\mathbf{V}^\perp, T^* \mathbb{G}_k(\mathbb{R}^n) \otimes \mathbf{V}) \cong \mathbf{V}^\perp \otimes (T^* \mathbb{G}_k(\mathbb{R}^n) \otimes \mathbf{V})$$

(identifying $\mathbf{V}^\perp \cong (\mathbf{V}^\perp)^*$ as usual). It turns out that this section determines an immersion of \mathbf{V}^\perp as a subbundle of $T^* \mathbb{G}_k(\mathbb{R}^n) \otimes \mathbf{V}$; we shall return to this question later in the Section.

We use the standard connections and tensors previously introduced in order to construct new differential operators on the tautological bundle \mathbf{V} and on its orthogonal complement \mathbf{V}^\perp . First of all, given an element $A \in \mathbb{R}^n$ we can associate to it two sections of the bundles \mathbf{V} and \mathbf{V}^\perp just using the projections: $s_A = \pi A$ and $s_A^\perp = \pi^\perp A$ with $A = s_A + s_A^\perp$; as A is constant,

$$0 = dA = ds_A + ds_A^\perp$$

so that

$$ds_A = -ds_A^\perp;$$

in the language already deployed

$$\nabla^{\mathbf{V}} s_A = \pi ds_A = -\pi ds_A^\perp = -II^\perp s_A^\perp.$$

These equations imply that

$$ds_A = -II^\perp s_A^\perp + II s_A. \quad (4)$$

For convenience we will combine the homomorphisms II and II^\perp to act upon any \mathbb{R}^n -valued function on $\mathbb{G}_3(\mathbb{R}^n)$, giving a mapping

$$i : C^\infty(\mathbb{G}_3(\mathbb{R}^n), \mathbb{R}^n) \longrightarrow \Gamma(T^* \otimes \mathbb{R}^n)$$

defined by

$$i(S) = II(\pi S) - II^\perp(\pi^\perp S). \quad (5)$$

in a way which is consistent with equation (4). Thus we have

$$ds_A = i(A) \quad (6)$$

and

$$ds_A^\perp = -i(A). \quad (7)$$

The image of II^\perp corresponds to elements of the type

$$\sum_{i=1}^k \lambda y \otimes v_i \otimes v_i \quad (8)$$

with $y \in \mathbf{V}^\perp$ and $\lambda \in \mathbb{R}$; this can be shown with the following argument: let us consider the decomposition as $SO(k) \times SO(n-k)$ modules of the involved bundles

$$\mathbf{V}^\perp \otimes \mathbf{V} \otimes \mathbf{V} \cong \mathbf{V}^\perp \otimes \mathbb{R} + \mathbf{V}^\perp \otimes (\mathbf{V} \otimes \mathbf{V})_0 \quad (9)$$

where $(\mathbf{V} \otimes \mathbf{V})_0$ is the tracefree part of the tensor product; Schur's Lemma guarantees that the second summand cannot contain any submodule isomorphic to \mathbf{V}^\perp , so the first summand consists of the unique submodule of this type in the right side term of (9). Therefore, as expression (8) provides an $SO(k) \times SO(n-k)$ -equivariant copy of \mathbf{V}^\perp inside this bundle, it must coincide with $II^\perp(\mathbf{V}^\perp)$. The same argument shows that

$$II(u) = \sum_{i=1}^{n-k} \lambda u \otimes w_i \otimes w_i$$

with $u \in \mathbf{V}$, $\lambda \in \mathbb{R}$. We want now to be more precise about these statements, and calculate explicitly the value of λ . This is done in the next proposition (in which tensor product symbols are omitted).

Proposition 2.1. *Let $A \in \mathbb{R}^n$ so that $A = u + y$ with $u \in V$ and $y \in V^\perp$ at the point V ; let v_j and w_i denote the elements of ON bases of V and V^\perp at V ; then*

$$II(u) = \sum_j u w_j w_j \quad (10)$$

and

$$II^\perp(y) = - \sum_i y v_i v_i. \quad (11)$$

Proof. We differentiate the section s_A along the curve $\alpha_{ij}(t)$ passing through V and with tangent vector $v_i w_j$ as in (3); let $u = \sum_{i=1}^k a_i v_i$ and $y = \sum_{j=1}^{n-k} b_j w_j$; then

$$\begin{aligned} s_A(\alpha_{ij})(t) &= a_1 v_1 + \cdots + \langle A, \cos r v_i + \sin r w_j \rangle (\cos r v_i + \sin r w_j) + \cdots + v_k \\ &= a_1 v_1 + \cdots + (a_i \cos r + b_j \sin r) (\cos r v_i + \sin r w_j) + \cdots + v_k \end{aligned}$$

so that

$$\frac{d}{dr} s_A(\alpha_{ij})(r)|_{r=0} = d s_A \cdot v_i w_j = b_j v_i + a_i w_j;$$

therefore, as an \mathbb{R}^n -valued 1-form,

$$\begin{aligned} d s_A &= \sum_{ij} b_j v_i v_i w_j + a_i w_j v_i w_j \\ &= \sum_i y v_i v_i + \sum_j u w_j w_j, \end{aligned}$$

where the second summand belongs to $\mathbf{V} \otimes \mathbf{V}^\perp \otimes \mathbf{V}^\perp$ and coincides with $II(u)$ as claimed. An analogous calculation for s_A^\perp gives

$$d s_A^\perp = - \sum_i y v_i v_i - \sum_j u w_j w_j$$

as expected from equation (7). ■

Observation. The opposite signs in (10) and (11) are consistent with the equation

$$0 = d \langle s_A, s_A^\perp \rangle|_V = \langle II(u), y \rangle + \langle u, II^\perp(y) \rangle$$

which expresses the fact that II and II^\perp are adjoint linear operators.

Proposition 2.1 shows that $\nabla^{\mathbf{V}} s_A$ is of the form seen in (8), or alternatively that if we call π_2 the projection on the second summand in the decomposition (9) and define $D \equiv \pi_2 \circ \nabla^{\mathbf{V}}$, the section s_A satisfies the *twistor-type* equation

$$D s_A = 0. \tag{12}$$

Symmetrically we can define another operator D^\perp such that

$$D^\perp s_A^\perp = 0. \tag{13}$$

Let us choose an orthonormal basis e_1, \dots, e_n of \mathbb{R}^n , every section S of the flat bundle $\mathbb{G}_k(\mathbb{R}^n) \times \mathbb{R}^n$ is nothing else than an n -tuple of functions

$$f_j : \mathbb{G}_k(\mathbb{R}^n) \longrightarrow \mathbb{R}^n$$

so that

$$S = \sum f_j e_j;$$

applying the exterior derivative on \mathbb{R}^n (which is a connection on the flat bundle) we obtain

$$dS = \sum df_j \otimes e_j$$

and if $1 \wedge i$ denotes an element in $\text{Hom}(T^* \otimes \mathbb{R}^n, (\otimes^2 T^*) \otimes \mathbb{R}^n)$ (where $T^* = T^*\mathbb{G}_k(\mathbb{R}^n)$ to lighten the notation) acting in the obvious way, we obtain

$$1 \wedge i(dS) = \sum df_j \wedge i(e_j);$$

on the other hand

$$d \sum f_j i(e_j) = \sum df_j \wedge i(e_j) + f_j di(e_j),$$

so if we can show that

$$di(e_j) = 0 \quad \forall j$$

we obtain the commutativity of the following diagram:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{d} & T^* \otimes \mathbb{R}^n \\ \downarrow i & & \downarrow 1 \wedge i \\ \mathbb{R}^n & \xrightarrow{d} & T^* \otimes \mathbb{R}^n \xrightarrow{d} \Lambda^2 T^* \otimes \mathbb{R}^n \end{array} \quad ; \quad (14)$$

but equation (6) implies:

$$di(e_j) = dds_{e_j} = 0,$$

because the e_j are constant. A consequence of Proposition 2.1 is that i is an injective map (because II and II^\perp are); if we can show that also $1 \wedge i$ is injective (and it happens to be in most part of cases, as we will see) looking at diagram (14) we can deduce the following facts: if $s \in \Gamma(\mathbf{V})$ satisfies $Ds = 0$, then $ds = i(s + s')$ for some $s' \in \Gamma(\mathbf{V}^\perp)$; this follows by comparing

$$ds = \nabla s + II(s)$$

with (5) and noting that $\pi s = s$ in this case: then $s' = -(II^\perp)^{-1}(\nabla s)$. Obviously $dds = 0$, so $d(s + s') = 0$ too, hence it is a constant element $A \in \mathbb{R}^n$. This implies the main result of this Section:

Theorem 2.2. *A section $s \in \Gamma(\mathbf{V})$ satisfies the twistor equation $Ds = 0$ if and only if exists another section $s' \in \Gamma(\mathbf{V}^\perp)$ such that $s + s' = A$ is a constant section of \mathbb{R}^n , provided $k > 1$ and $n - k > 1$.*

In other words sections of type s_A are the only solutions of equation (12), under these hypotheses.

The missing piece to prove Theorem 2.2 is injectivity of $1 \wedge i$. To prove that we start defining another map:

$$c : \Gamma(T^* \otimes \mathbb{R}^n) \longrightarrow \Gamma(\mathbb{R}^n)$$

acting as a contraction in the following way:

$$c\left(\sum_{ijk} a^{ijk} v_i w_j v_k + \sum_{lmo} b^{lmo} w_l v_m w_o\right) = \sum_{ij} a^{iji} w_j + \sum_{lm} b^{lml} v_m.$$

The same map acts also on $\tau \in (\otimes^q T^*) \otimes \mathbb{R}^n$ in the following way: if $\tau = \tau' \otimes \theta$ with $\tau' \in \otimes^{q-1} T^*$ and $\theta \in T^* \otimes \mathbb{R}^n$ then

$$c(\tau) = \tau' \otimes c(\theta)$$

and then extending linearly.

We are now in position to prove the previously stated assertion, which concludes the proof of Theorem 2.2:

Lemma 2.3. *The map $1 \wedge i$ is injective, provided $k > 1$ and $n - k > 1$.*

Proof. Given two bases v^i of V and w^j of V^\perp an element in $T^* \otimes \mathbb{R}^n$ is described by

$$\tau = \sum_{ijh} a^{ijh} v_i w_j v_h + \sum_{lmo} b^{lmo} v_l w_m w_o;$$

now we will prove that $c \circ 1 \wedge i$ is injective, so that $1 \wedge i$ must be. So we get

$$\begin{aligned} 1 \wedge i(\tau) &= \sum_{ijh\mu} a^{ijh} (v_i w_j \wedge v_h w_\mu) w_\mu + \sum_{lmo\nu} b^{lmo} (v_l w_m \wedge w_o v_\nu) v_\nu \\ &= \sum_{ijh\mu} a^{ijh} (v_i w_j \otimes v_h w_\mu - v_h w_\mu \otimes v_i w_j) w_\mu \\ &\quad + \sum_{lmo\nu} b^{lmo} (v_l w_m \otimes w_o v_\nu - v_o v_\nu \otimes w_l v_m) v_\nu \end{aligned}$$

and applying the contraction

$$\begin{aligned} c(1 \wedge i(\tau)) &= \sum_{ijh\mu} a^{ijh} (v_i w_j \otimes v_h - v_h w_\mu \otimes v_i \delta_\mu^j) \\ &\quad + \sum_{lmo\nu} b^{lmo} (v_l w_m \otimes w_o - v_o w_\nu \otimes w_l \delta_\nu^m). \end{aligned}$$

Now imposing that it's zero, we get the following couples of equations:

$$\begin{cases} (n - k) a^{ijh} - a^{hji} = 0 \\ (n - k) a^{hji} - a^{ijh} = 0 \end{cases}$$

and

$$\begin{cases} k b^{lmo} - b^{oml} = 0 \\ k b^{oml} - b^{lmo} = 0 \end{cases}$$

which imply

$$(n - k)^2 a^{ijh} = a^{ijh}$$

and

$$k^2 b^{lmo} = b^{lmo}$$

which are absurd if $k > 1$ and $n - k > 1$. ■

3 The two twistor equations

Let us consider a compact Lie group G acting by isometries on a QK manifold M ; then its moment map μ can be described locally as

$$\mu = \sum_{i=1}^3 \omega_i \otimes B_i \tag{15}$$

with ω_i a local orthonormal basis for S^2H and B_i belonging to \mathfrak{g} . Suppose that $V := \text{span}\{B_1, B_2, B_3\}$ is a 3-dimensional subspace of \mathfrak{g} : then V is independent of the trivialization, as the structure group of S^2H is $SO(3)$. Therefore we obtain a well defined map

$$\Psi : M_0 \longrightarrow \mathbb{G}_3(\mathfrak{g})$$

where $M_0 \subset M$ is defined as the subset where $V(x)$ is 3-dimensional; this turns out to be an open dense subset of the union $\bigcup S$ of G -orbits S on M such that $\dim S \geq 3$ ([23, Proposition 3.5]). Therefore if the dimension of the maximal G orbits in M is big enough, then M_0 is an open dense subset of M .

Assumption. From now on we will assume that

$$B_i = \lambda(x) v_i \tag{16}$$

for v_i an orthonormal basis of V . This hypothesis is not excessively restrictive, in the sense that it is compatible with the existence of open $G_{\mathbb{C}}$ orbits on the twistor space $\mathcal{Z} = \mathbb{P}(\mathcal{U})$: in fact the projectivization of the complex-contact moment map f induced on \mathcal{Z} satisfies

$$(\mathbb{P}f)(\omega_1) = \text{span}_{\mathbb{C}}\{B_2 + \iota B_3\},$$

and in this case this turns out to be a ray of nilpotent elements in $\mathfrak{g}_{\mathbb{C}}$ (see ([23, §3]). Nilpotent elements belong to the zero set of any invariant symmetric tensor over $\mathfrak{g}_{\mathbb{C}}$, in particular with respect to the Killing form: for by Engel's

theorem their adjoint representation can be given in terms of strictly upper triangular matrices, with respect to a suitable basis, and the product of such matrices is still strictly upper triangular and hence traceless; in other words

$$\begin{aligned} 0 &= \text{Tr}(ad_{B_2 + \iota B_3} \circ ad_{B_2 + \iota B_3}) = \langle B_2 + \iota B_3, B_2 + \iota B_3 \rangle \\ &= \|B_1\|^2 - \|B_2\|^2 + 2\iota \langle B_2, B_3 \rangle, \end{aligned}$$

which implies $B_2 \perp B_3$ and $\|B_2\| = \|B_3\|$, conditions that are equivalent to the assumption, permuting cyclically the indices. Therefore condition (16) holds for all unstable manifolds described in [23], as in that case the twistor bundle \mathcal{Z} is $G_{\mathbb{C}}$ -homogeneous. We assume throughout the Section that this condition holds for the moment map μ .

Using the map Ψ , we can construct on M_0 the pullback bundle $\Psi^*(\mathbf{V})$; the latter is unique up to isomorphism of bundles (see [24, Chap.I, Prop. 2.15]). More precisely, any vector bundle $W \rightarrow M_0$ for which there exists a map of bundles $\hat{\Phi} : W \rightarrow \mathbf{V}$ which is injective on the fibres, and a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\hat{\Phi}} & \mathbf{V} \\ p_V^* \downarrow & & \downarrow p_V \\ M_0 & \xrightarrow{\Psi} & \mathbb{G}_3(\mathfrak{g}), \end{array} \quad (17)$$

is necessarily isomorphic to $\Psi^*(\mathbf{V})$.

Lemma 3.1. *We have the following isomorphism of bundles on M_0 :*

$$S^2 H \cong \Psi^*(\mathbf{V}).$$

Proof. To complete the commutative diagram (17), define the morphism of bundles

$$\hat{\Phi} : S^2 H \longrightarrow \mathbf{V}$$

by

$$(x, \omega_i(x)) \longmapsto (\text{span}\{B_1(x), B_2(x), B_3(x)\}, B_i(x))$$

(see (15)), extending linearly on the fibres. This corresponds to the contraction of a vector $v \in S^2 H_x$ with the $S^2 H$ component of $\mu(x)$ using the metric, so it does not depend on the trivialization (the structure group preserves the metric) and is injective on the fibres by definition of M_0 . ■

We should point out that $\hat{\Phi}$ is not an isometry of Riemannian bundles in general; nevertheless under the hypotheses discussed above, we can assume that $\hat{\Phi}$ is a conformal map of Riemannian bundles, considering $S^2 H$ and \mathbf{V} to be equipped with the natural metrics coming respectively from M and from $\mathbb{G}_3(\mathfrak{g})$.

Let us now recall some well-known differential operators (the symbol Γ denoting space of sections is omitted): the *Dirac operator*

$$\delta : S^2 H \xrightarrow{\nabla} E \otimes H \otimes S^2 H \hookrightarrow (E \otimes \underline{H}) \otimes (H \otimes \underline{H}^*) \longrightarrow T^*$$

where the underlined terms are contracted and $T^* = E \otimes H$; the *QK twistor operator* is defined as follows:

$$\mathcal{D} : S^2 H \xrightarrow{\nabla} E \otimes H \otimes S^2 H \xrightarrow{\text{sym}} E \otimes S^3 H ,$$

where we symmetrize after covariant differentiation. In [19, Lemma 6.5], under the assumption of nonzero scalar curvature, Salamon proved that sections of $S^2 H$ belonging to $\ker \mathcal{D}$ are in bijection with the elements in the space \mathcal{K} of Killing vector fields preserving the QK structure; this means that if ν is in $\ker \mathcal{D}$ then $\delta(\nu)$ is dual to a Killing vector field $\tilde{A} \in \mathcal{K}$, and on the other hand $\nu = \mu_A$, or in other words

$$\mathcal{D} \mu_A = 0 \tag{18}$$

and all elements in $\ker \mathcal{D}$ are of this form.

Recall now what was discussed for Grassmannians in Section 2: there we introduced another differential operator D on the tautological bundle \mathbf{V} over $\mathbb{G}_3(\mathfrak{g})$; the elements in its kernel were proved to be precisely the sections s_A obtained by projection from the trivial bundle with fibre \mathfrak{g} (see Theorem 2.2). We want to relate the kernels of \mathcal{D} and D through the map Ψ induced by μ ; recall that the bundle homomorphism $\hat{\Phi}$ is defined up to a bundle automorphism of $S^2 H$; we can for instance introduce a dilation

$$\xi(x, w) = (x, \frac{w}{\|B_i\|}), \tag{19}$$

which is independent of the trivialization; in this way

$$\hat{\Xi}(\omega_i) := \hat{\Phi} \circ \xi(\omega_i) = \frac{B_i}{\|B_i\|},$$

and so an orthonormal basis is sent to another orthonormal basis: this is therefore an isometry of the two bundles compatible with the map Ψ induced by μ .

We can now state the main result of this Section. Let us denote by $\mathcal{K}_{\mathfrak{g}} \subset \mathcal{K}$ the subspace of Killing vector fields induced by \mathfrak{g} and by $(\ker \mathcal{D})_{\mathfrak{g}}$ the space of the corresponding twistor sections; then

Proposition 3.2. *There exists a lift $\hat{\Psi}$ of the map Ψ such that*

$$\hat{\Psi}(\mu_A) = s_A ,$$

inducing a bijective linear map

$$(\ker \mathcal{D})_{\mathfrak{g}} \longrightarrow \ker D .$$

Proof. We are looking for a lift $\hat{\Psi}$ such that the diagram

$$\begin{array}{ccc} S^2H & \xrightarrow{\hat{\Psi}} & \mathbf{V} \\ \mu_A \uparrow & & \uparrow s_A \\ M_0 & \xrightarrow{\Psi} & \mathbb{G}_3(\mathfrak{g}) . \end{array}$$

commutes; recall the usual local description (15) of μ , and let us define $\hat{\Psi}$ so that

$$\hat{\Psi}(\omega_i) = \frac{B_i}{\|B_i\|^2} ,$$

obtained by composing $\hat{\Phi}$ with the dilation ξ^2 (see (19)); this is again a lift of Ψ ; consider as usual $\mu_A \in \Gamma(S^2H)$ satisfying the twistor equation; then

$$\begin{aligned} \hat{\Psi}(\mu_A) &= \hat{\Psi}\left(\sum_i \omega_i \langle B_i, A \rangle\right) \\ &= \sum_i \frac{B_i}{\|B_i\|^2} \langle B_i, A \rangle \\ &= \pi_V A = s_A , \end{aligned}$$

as required. As the lift $\hat{\Psi}$ is injective on the fibres, and as

$$\dim(\ker \mathcal{D})_{\mathfrak{g}} = \dim \mathcal{K}_{\mathfrak{g}} = \dim \mathfrak{g} = \dim \ker D$$

the last assertion follows. ■

The situation can be summarized in diagram (20):

$$\begin{array}{ccc} & \boxed{A \in \mathfrak{g}} & \\ & \swarrow \quad \searrow & \\ \boxed{s_A \in \ker D} & \text{---} & \boxed{\mu_A \in (\ker \mathcal{D})_{\mathfrak{g}}} \end{array} \tag{20}$$

Observation. We can interpret μ as a collection of $n = \dim \mathfrak{g}$ sections of S^2H : if A_i are an orthonormal basis for \mathfrak{g} the moment map μ is completely determined by the μ_{A_i} . Locally we get

$$B_i = \sum_j a_i^j A_j \tag{21}$$

so that

$$\mu_{A_i} = \sum_j a_i^j \omega_j.$$

For instance, if a section $\nu \in \Gamma(S^2H)$ is given locally by

$$\nu = \sum_i c^i \omega_i$$

then

$$\hat{\Phi}(\nu) = \sum_i c^i B_i;$$

with respect to the basis A_i of \mathfrak{g} the local description of the morphism $\hat{\Phi}$ is encoded in the $(3 \times (n-3))$ matrix of the coefficients a_j^i seen in (21).

4 The $Sp(1)Sp(n)$ structure

We are going now to introduce an alternative description of the endomorphisms I_1, I_2, I_3 in a purely algebraic setting, using the holonomy representation at a fixed point $x \in M$.

Let h, \hat{h} denote a unitary basis of H , in such a way that $\omega_H(h, \hat{h}) = 1$; with respect to this basis we have

$$\omega_H = h \wedge \hat{h} = \frac{1}{2}(h\hat{h} - \hat{h}h). \quad (22)$$

We can in terms of h, \hat{h} determine a basis of S^2H :

$$\begin{aligned} I_1 &= \iota(h \vee \hat{h}) \\ I_2 &= h^2 + \hat{h}^2 \\ I_3 &= \iota(h^2 - \hat{h}^2) \end{aligned} \quad (23)$$

are orthogonal of norm $\sqrt{2}$ with respect to the metric $\omega_H \otimes \omega_H$ induced on S^2H ; they satisfy the same relations of quaternions:

$$I_k^2 = -1 \quad , \quad I_i I_j = \text{sgn}_{(ijk)} I_k$$

with $\text{sgn}_{(ijk)}$ the sign of the permutation; the composition is obtained by contracting again with ω_H .

Consider now the case where the $Sp(1)$ representation inside $Sp(1)Sp(n)$ is such that the projection on the $Sp(n)$ factor is nonzero: this means that the E representation is nontrivial under this $Sp(1)$ action.

In this case it is significant to analyze the quaternionic action from the point of view of these new $Sp(1)$ representations. First we adopt the following notation: we have the symmetrization map S acting on tensors as

$$S(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\pi^n} x_{\pi^n(1)} \otimes \cdots \otimes x_{\pi^n(n)}$$

where π^n varies in the group of permutations on n elements; the map extends linearly. We give then the following definition: we denote as

$$\{\cdot, \cdot\} : \Sigma^k \otimes \Sigma^h \longrightarrow \Sigma^{h+k}$$

the symmetrization of the two factors, more explicitly if

$$\alpha = \sum_{\pi^k} \alpha_{\pi^k(1)} \otimes \cdots \otimes \alpha_{\pi^k(k)} \in \Sigma^k \quad , \quad \beta = \sum_{\pi^h} \beta_{\pi^h(1)} \otimes \cdots \otimes \beta_{\pi^h(h)} \in \Sigma^h$$

then

$$\{\alpha \otimes \beta\} = \sum_{\pi^k, \pi^h} S(\alpha_{\pi^k(1)} \otimes \cdots \otimes \alpha_{\pi^k(k)} \otimes \beta_{\pi^h(1)} \otimes \cdots \otimes \beta_{\pi^h(h)}) .$$

In particular we denote by σ the map $\{\cdot, \cdot\}$ when the first index is 1:

$$\sigma := \{\cdot, \cdot\} : \Sigma^1 \otimes \Sigma^i \longrightarrow \Sigma^{i+1} . \quad (24)$$

Consider now for simplicity the case that E corresponds to an irreducible $Sp(1)$ representation; then

$$T_x M_{\mathbb{C}} \cong \Sigma^1 \otimes \Sigma^{i-1}$$

and using Clebsch-Gordan relation, we obtain

$$T_x M_{\mathbb{C}} \cong \Sigma^i + \Sigma^{i-2} \hookrightarrow \Sigma^{i+2} + \Sigma^i + \Sigma^{i-2} \cong \Sigma^2 \otimes \Sigma^i ; \quad (25)$$

more precisely $T_x M_{\mathbb{C}}$ coincides with the kernel of the symmetrization

$$\{\cdot, \cdot\} : \Sigma^2 \otimes \Sigma^i \longrightarrow \Sigma^{i+2} .$$

Example. There are (up to conjugation) three non-trivial homomorphisms $Sp(1) \rightarrow Sp(2)$: two correspond to the roots, but in these cases the decomposition of the standard $Sp(2)$ representation \mathbb{C}^4 is not irreducible; in fact

$$E = \mathbb{C}^4 = \Sigma^0 + \Sigma^0 + \Sigma^1$$

for the long root, and comparing with the known decomposition of the adjoint representation one has

$$\mathfrak{sp}(2) = S^2(\mathbb{C}^4) = S^2(2\Sigma^0 + \Sigma^1) = \Sigma^2 + 2\Sigma^1 + 3\Sigma^0 ;$$

for the short root we have instead

$$E = \mathbb{C}^4 = \Sigma^1 + \Sigma^1$$

as in fact

$$\mathfrak{sp}(2) = S^2(\mathbb{C}^4) = S^2(2\Sigma^1) = 3\Sigma^2 + \Sigma^0 .$$

There is a third embedding, corresponding to the $\mathfrak{sl}(2, \mathbb{C})$ triple

$$X = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

obtained using the recipe in [7], for which

$$E = \mathbb{C}^4 = \Sigma^3. \quad (26)$$

Observation. This last can be interpreted in the following way: recall that the decomposition of the Lie algebra \mathfrak{g}_2 with respect to $\mathfrak{so}(4) \subset \mathfrak{g}_2$ is given by

$$\Sigma_+^2 + \Sigma_-^2 + \Sigma_-^1 \otimes \Sigma_+^3,$$

where Σ_{\pm}^k denote the representations of the $\mathfrak{sp}(1)$ corresponding to the long (+) or to the short(-) root; so considering the diagonal embedding

$$\mathfrak{sp}(1)_{\Delta} \hookrightarrow \mathfrak{so}(4) = \mathfrak{sp}(1)_+ + \mathfrak{sp}(1)_- \hookrightarrow \mathfrak{sp}(1)_+ + \mathfrak{sp}(2),$$

consistently with the $Sp(1)Sp(2)$ structure of the Wolf space

$$\frac{G_2}{SO(4)},$$

we have a description of its tangent space in the EH formalism as $H \otimes E \cong \Sigma^1 \otimes \Sigma^3$, corresponding to the representation in (26).

The action of $S^2H \cong \Sigma^2$ on $T_x M_{\mathbb{C}}$ can be therefore expressed suitably exploiting this new formulation, involving the Σ^2 factor instead of the $\Sigma^1 = H$; to understand more deeply this Σ^2 -approach we need to define more explicitly the invariant immersion (25). Let us define the map as

$$Q : \Sigma^1 \otimes \Sigma^{i-1} \xrightarrow{\omega_H \otimes} \underline{\Sigma^1} \otimes \underline{\Sigma^1} \otimes \underline{\Sigma^1} \otimes \underline{\Sigma^{i-1}} \xrightarrow{\{\cdot, \cdot\}} \Sigma^2 \otimes \Sigma^i \quad (27)$$

acting in the following way: if

$$Y = h \otimes \beta + \hat{h} \otimes \hat{\beta} \in \Sigma^1 \otimes \Sigma^{i-1}, \quad \beta, \hat{\beta} \in \Sigma^{i-1}$$

then

$$Q(Y) = \frac{1}{2}\{hh\}\{\hat{h}\beta\} + \frac{1}{4}(h\hat{h} + \hat{h}h)(\{\hat{h}\hat{\beta}\} - \{h\beta\}) - \frac{1}{2}\{\hat{h}\hat{h}\}\{h\hat{\beta}\} \quad (28)$$

is obtained, after tensorizing with the invariant element ω_H , by symmetrization of the tensorial factors in accordance with the simple or double underlining marks in (27).

Our next aim is to express the quaternionic action in terms of this description: a first guess in this sense is that for $Q(Y) = \sum v_i \otimes p_i$ then

$$Q(I_1 Y) = v_2 \otimes p_3 + v_3 \otimes p_2 ,$$

mimicking the adjoint representation of $\mathfrak{su}(2)$ on itself; but this is not correct, as at the second step

$$Q(I_1^2 Y) = -v_2 \otimes p_2 - v_3 \otimes p_3 ,$$

which is not $-Id$. Something more is needed to “reconstruct” the missing term $-v_1 \otimes p_1$.

The next Proposition gives the correct answer in order to express the quaternionic action from the Σ^2 viewpoint:

Proposition 4.1. *Let $Y \in T_x M = \Sigma^1 \otimes \Sigma^i$; if $Q(Y) = \sum v_i \otimes p_i$ then*

$$Q(I_1 Y) = v_1 \otimes \frac{1}{4}\sigma(Y) + v_2 \otimes p_3 - v_3 \otimes p_2 . \quad (29)$$

Proof. We have the definition of $Q(Y)$ as in (28): then if we identify v_i with the basis I_i defined in (23), grouping the terms properly we obtain

$$\begin{aligned} p_1 &= -\frac{i}{4}(\{\hat{h}\hat{\beta}\} - \{h\beta\}) \\ p_2 &= \frac{1}{4}(\{\hat{h}\beta\} - \{h\hat{\beta}\}) \\ p_3 &= -\frac{i}{4}(\{\hat{h}\beta\} + \{h\hat{\beta}\}) ; \end{aligned}$$

the quaternionic action of I_1 on Y is given, in the Σ^1 context, by

$$I_1 Y = -ih \otimes \beta + i\hat{h} \otimes \hat{\beta} ;$$

so we obtain

$$Q(I_1 Y) = -\frac{i}{2}\{hh\}\{\hat{h}\beta\} + \frac{i}{4}(h\hat{h} + \hat{h}h)(\{h\beta\} + \{\hat{h}\hat{\beta}\}) - \frac{i}{2}\{\hat{h}\hat{h}\}\{h\hat{\beta}\}$$

and in the form $Q(I_1 Y) = \sum_{i=1}^3 v_i \otimes q_i^1$ we have

$$\begin{aligned} q_1^1 &= \frac{i}{4}(\{h\beta\} + \{\hat{h}\hat{\beta}\}) \\ q_2^1 &= -\frac{i}{4}(\{\hat{h}\beta\} + \{h\hat{\beta}\}) \\ q_3^1 &= -\frac{1}{4}(\{\hat{h}\beta\} - \{h\hat{\beta}\}) ; \end{aligned}$$

the conclusion follows by the definition of σ (24) and comparing the two sets of equalities. ■

In the same way we obtain for the other quaternionic elements

$$\begin{aligned} I_2 Y &= -\hat{h} \otimes \beta + h \otimes \hat{\beta} \\ I_3 Y &= \imath \hat{h} \otimes \beta + \imath h \otimes \hat{\beta} \end{aligned}$$

so that

$$\begin{aligned} Q(I_2 Y) &= \frac{1}{2} \{hh\} \{\hat{h}\hat{\beta}\} - \frac{1}{2} \{h\hat{h}\} (\{\hat{h}\beta\} + \{h\hat{\beta}\}) + \frac{1}{2} \{\hat{h}\hat{h}\} \{h\beta\} \\ Q(I_3 Y) &= \frac{\imath}{2} \{hh\} \{\hat{h}\hat{\beta}\} + \frac{\imath}{2} \{h\hat{h}\} (\{\hat{h}\beta\} - \{h\hat{\beta}\}) - \frac{\imath}{2} \{\hat{h}\hat{h}\} \{h\beta\} \end{aligned}$$

which imply the equalities

$$q_j^i = \eta_{ijk} p_k - \delta_i^j \frac{1}{4} \sigma(Y),$$

where $\eta_{ijk} = \text{sgn}_{ijk}$ if $i \neq j$, otherwise $\eta_{ijk} = 0$; moreover

$$p_i = -\frac{1}{4} \sigma(I_i Y).$$

We can therefore state the quaternionic relations in terms of this description: for example

$$\begin{aligned} Q(I_1^2 Y) &= Q(I_1 I_1 Y) = -v_1 \otimes \frac{1}{4} \sigma(I_1 Y) - v_2 \otimes p_2 - v_3 \otimes p_3 \\ &= -v_1 \otimes p_1 - v_2 \otimes p_2 - v_3 \otimes p_3 \\ &= -Q(Y) \end{aligned}$$

and also

$$\begin{aligned} Q(I_1 I_2 Y) &= -v_1 \otimes \frac{1}{4} \sigma(I_2 Y) - v_2 \otimes q_3^2 - v_3 \otimes q_2^2 \\ &= -v_1 \otimes p_2 + v_2 \otimes p_1 - v_3 \otimes \frac{1}{4} \sigma(Y) \\ &= Q(I_3 Y) \end{aligned}$$

as expected.

5 The Coincidence Theorem

Another way of expressing the *twistor equation* (1) is given by

$$\nabla^{S^2 H} \mu_A = k \sum_{i=1}^3 I_i \tilde{A}^b \otimes I_i, \quad (30)$$

where \tilde{A} is the Killing vector field generated by A in \mathfrak{g} , the symbol \flat means passing to the corresponding 1-form via the metric and k is the scalar curvature, which is constant as the metric is Einstein (for simplicity we can put $k = 1$). On the other hand on \mathbf{V} we have defined the sections s_A and the natural connection $\nabla^{\mathbf{V}}$ so that (see (8) and Proposition 2.1)

$$\nabla^{\mathbf{V}} s_A = \sum_{i=1}^3 s_A^\perp \otimes v_i \otimes v_i.$$

In general, given a differentiable map $\Psi : M \rightarrow N$ of manifolds, and an isomorphism $\hat{\Psi}$ between vector bundles $E \rightarrow F$ on the manifold M and N respectively, the second one equipped with a connection ∇^F , we can define the *pullback connection* $\hat{\Psi}^* \nabla^F$ acting in the following way on elements s of $\Gamma(E)$:

$$(\Psi^* \nabla^F)_Y(s) := \hat{\Psi}^*(\nabla_{(\Psi_* Y)}^F(\hat{\Psi} s))$$

where $Y \in T_x M$ and $\hat{\Psi}^*$ means taking the pullback section.

We want to apply this construction in our case, with the map $\Psi : M \rightarrow \mathbb{G}_3(\mathfrak{g})$ induced by μ , $N = \mathbb{G}_3(\mathfrak{g})$, $E = S^2 H$, $F = \mathbf{V}$; our aim is to relate, at a fixed point $x \in M$, the action of the quaternionic structure on 1-forms induced by G (the duals of the Killing vector fields) with special cotangent vectors on the Grassmannian $\mathbb{G}_3(\mathfrak{g})$:

Lemma 5.1. *Let $M, \mathfrak{g}, \mathbb{G}_3(\mathfrak{g}), \mu$ be defined as usual, with*

$$\mu = \sum_{i=1}^3 I_i \otimes B_i$$

where $B_i = \lambda v_i$, λ a differentiable G -invariant function on M and v_i an orthonormal basis of a point $V \in \mathbb{G}_3(\mathfrak{g})$; let us choose $A \in V^\perp \subset \mathfrak{g}$; then at the point x such that $\Psi(x) = V$, for Ψ induced by μ as usual, we have

$$\frac{1}{\lambda} I_i \tilde{A}^\flat = \Psi^*(A \otimes v_i)^\flat, \quad (31)$$

where $A \otimes v_i \in T_x \mathbb{G}_3(\mathfrak{g})$. Moreover we have $\|\mu\|^2 = 3\lambda^2$.

Proof. Let Ψ denote the conformal lift of the map μ so that

$$\Psi(I_i) = \frac{1}{\lambda^2} B_i; \quad (32)$$

hence as seen in Proposition 3.2

$$\Psi(\mu_A) = s_A;$$

then applying the $\Psi^*\nabla^{\mathbf{V}}$ connection of S^2H to μ_A we obtain

$$\begin{aligned}
(\Psi^*\nabla^{\mathbf{V}})\mu_A &= \Psi^*(\nabla^{\mathbf{V}}(\Psi(\mu_A))) \\
&= \Psi^*(\nabla^{\mathbf{V}}s_A) \\
&= \Psi^*\left(\sum_{i=1}^3 s_A^\perp \otimes v_i \otimes v_i\right) \\
&= \lambda \sum_{i=1}^3 \Psi^*(s_A^\perp \otimes v_i) \otimes I_i;
\end{aligned} \tag{33}$$

on the other hand the difference of two connections on the same vector bundle is a tensor, so given any section $s \in S^2H$ which vanishes at a point $x \in M$

$$(\nabla^{S^2H} - \Psi^*\nabla^{\mathbf{V}})s(x) = 0.$$

This is precisely the case for the section μ_A at the point x for which $\Psi(S^2H_x) = V$, because $A \in V^\perp$ by hypothesis; in other words

$$\nabla^{S^2H}\mu_A|_x = (\Psi^*\nabla^{\mathbf{V}})\mu_A|_x.$$

In the light of the calculations in (33) and of the twistor equation (30), we can deduce

$$\sum_{i=1}^3 I_i \tilde{A}^\flat \otimes I_i = \lambda \sum_{i=1}^3 \Psi^*(s_A^\perp \otimes v_i) \otimes I_i;$$

the result follows considering that $s_A^\perp = A$ at V . ■

Lemma 5.1 leads to various ways of relating elements in the respective spaces $T_x M$ and $T_V \mathbb{G}_3(\mathfrak{g})$ and the quaternionic elements I_i ; nevertheless it is stated merely in terms of 1-forms, whereas we are interested in involving the two metrics in this interplay. To this aim, let us define a linear transformation \sharp of $T_x M$ by

$$(X)^\sharp := (\Psi^*((\Psi_* X)^\flat))^\sharp \tag{34}$$

in $\text{End}(T_x M)$. This corresponds to moving in a counterclockwise sense around the following diagram, starting from bottom left:

$$\begin{array}{ccc}
T_x^* M & \xleftarrow{\Psi^*} & T_V^* \mathbb{G}_3 \\
\downarrow \sharp & & \uparrow \flat \\
T_x M & \xrightarrow{\Psi_*} & T_V \mathbb{G}_3
\end{array} . \tag{35}$$

Thus the linear endomorphism $(\cdot)^\sharp$ measures the noncommutativity of the diagram (35), and the difference between the pullbacked Grassmannian metric from the quaternionic one.

We are in position now to prove the *Coincidence Theorem*:

Theorem 5.2. *Let $Y \in T_x M$ such that*

$$\Psi_* Y = \sum v_i \otimes p_i ;$$

for $p_i \in V^\perp$ with $V = \Psi(x)$; then

$$(Y)^\natural = \frac{1}{\lambda} \sum_i I_i \tilde{p}_i.$$

Proof. Using the definitions and (31) we obtain

$$\begin{aligned} (\Psi_* Y)^\flat(\Psi_* Z) &= \langle \sum v_i \otimes p_i, \Psi_* Z \rangle_{\mathbb{G}_3} \\ &= \frac{1}{\lambda} \langle \sum I_i \tilde{p}_i, Z \rangle_M \end{aligned}$$

for any $Z \in T_x M$, hence the conclusion. ■

Theorem 5.2 provides a memorable way of “converting” tangent vectors of $\mathbb{G}_3(\mathfrak{g})$ to tangent vectors on M by means of the correspondence

$$\begin{aligned} v_i &\longrightarrow I_i \\ p_i &\longrightarrow \tilde{p}_i \end{aligned}$$

for $p_i \in V^\perp$.

The equivariance of the moment map μ implies that Killing vector fields on M map to Killing vector fields on $\mathbb{G}_3(\mathfrak{g})$: in other words if \tilde{A} is induced by $A \in \mathfrak{g}$ on M , then

$$\Psi_* \tilde{A} = \sum_{i=1}^3 v_i \otimes [A, v_i]^\perp.$$

Let now $\alpha = (\sum_{i=1}^3 v_i \otimes p_i)^\flat \in T_x^* \mathbb{G}_3(\mathfrak{g})$ and let A_r be an orthonormal basis of V^\perp ; then

$$\begin{aligned} \sum_{r=1}^{n-3} \langle \Psi^* \alpha, \tilde{A}_r \rangle A_r &= \sum_{r=1}^{n-3} \langle \alpha, \Psi_* \tilde{A}_r \rangle A_r = \sum_{i,r} \langle p_i, [v_i, A_r]^\perp \rangle A_r \\ &= \sum_{i,r} \langle p_i, [v_i, A_r] \rangle A_r = \sum_{i,r} \langle [p_i, v_i], A_r \rangle A_r \\ &= \sum_i [p_i, v_i]^\perp. \end{aligned}$$

We can therefore define a mapping

$$\rho : T_x^* M \longrightarrow V^\perp \tag{36}$$

by $\rho(\zeta) = \sum_r \langle \zeta, \tilde{A}_r \rangle A_r$; so if $\alpha \in T_x^* \mathbb{G}_3(\mathfrak{g})$, then $\Psi^* \alpha \in T_x^* M$, and the composition $\tilde{\gamma} = \rho \circ \Psi^*$ is a map

$$\tilde{\gamma} : T_x^* \mathbb{G}_3(\mathfrak{g}) \longrightarrow V^\perp .$$

defined by $\tilde{\gamma}(\alpha) = \sum_i [v_i, p_i]^\perp$; this operator can be described as

$$\tilde{\gamma} = \pi^\perp \circ \gamma$$

where $\gamma(\alpha) = \sum_i [v_i, p_i]$ is the obstruction to the orthogonality of α to the G -orbit: in fact

Lemma 5.3. *A tangent vector $P = \sum_{i=1}^3 v_i \otimes p_i \in T_V \mathbb{G}_3(\mathfrak{g})$ is orthogonal to the G -orbit through the point V if and only if $\gamma(P) = 0$.*

Proof. For any $A \in \mathfrak{g}$ let us consider the Killing vector field \tilde{A} on $\mathbb{G}_3(\mathfrak{g})$; the condition of orthogonality of P is expressed by

$$\begin{aligned} 0 &= \langle \tilde{A}, P \rangle = \sum_{i=1}^3 \langle [A, v_i]^\perp, p_i \rangle = \\ &= \sum_{i=1}^3 \langle [A, v_i], p_i \rangle = \sum_{i=1}^3 \langle A, [v_i, p_i] \rangle = \\ &= \langle A, \gamma(P) \rangle \quad . \quad \blacksquare \end{aligned}$$

We give now a more explicit description of the quaternionic endomorphisms:

Proposition 5.4. *Let $Y \in T_x M$ so that*

$$\Psi_* Y = v_1 \otimes p_1 + v_2 \otimes p_2 + v_3 \otimes p_3 ;$$

then we have

$$\Psi_* I_1 Y = \frac{1}{\lambda} v_1 \otimes \rho(Y^b) - v_2 \otimes p_3 + v_3 \otimes p_2 . \quad (37)$$

Proof. Consider any $A \in V^\perp$, then

$$\begin{aligned} \langle p_1, A \rangle_K &= \langle \Psi_* Y, A \otimes v_1 \rangle_{\mathbb{G}_3} = \frac{1}{\lambda} \langle I_1 \tilde{A}^b, Y \rangle \\ &= \frac{1}{\lambda} \langle I_1 \tilde{A}, Y \rangle_M = -\frac{1}{\lambda} \langle \tilde{A}, I_1 Y \rangle_M \\ &= -\frac{1}{\lambda} \langle I_1 Y^b, \tilde{A} \rangle , \end{aligned} \quad (38)$$

where $\langle, \rangle_{M, \mathbb{G}}$ denote the respective Riemannian metrics, \langle, \rangle_K minus the Killing form on \mathfrak{g} and \langle, \rangle without subscript is merely the contraction of a cotangent and tangent vector; then considering (38) and (36)

$$\begin{aligned} p_1 &= \sum_r \langle p_1, A_r \rangle_K A_r = -\frac{1}{\lambda} \sum_r \langle I_1 Y^\flat, \tilde{A}_r \rangle A_r \\ &= -\frac{1}{\lambda} \rho(I_1 Y^\flat) \end{aligned}$$

and analogously

$$p_i = -\frac{1}{\lambda} \rho(I_i Y^\flat), \quad i = 2, 3;$$

in consequence

$$\begin{aligned} \Psi_* I_1 Y &= \frac{1}{\lambda} v_1 \otimes \rho(Y^\flat) - \frac{1}{\lambda} v_2 \otimes \rho(I_3 Y^\flat) + \frac{1}{\lambda} v_3 \otimes \rho(I_2 Y^\flat) \\ &= \frac{1}{\lambda} v_1 \otimes \rho(Y^\flat) - v_2 \otimes p_3 + v_3 \otimes p_2. \blacksquare \end{aligned}$$

Clearly analogous assertions are valid for I_2 and I_3 .

Remark. Assuming that Ψ_* is injective at the point x , we can define the *push forward* of the endomorphisms I_k in the obvious way, namely via the equation:

$$(\Psi_* I_k) Z := \Psi_*(I_k(\Psi_*^{-1} Z))$$

and Proposition 5.4. A striking feature of (37) is that in the expression obtained the first summand is independent from I_1 . The operators ρ, γ appear as the essential ingredient to reconstruct the quaternionic action; the other summands $-v_2 \otimes p_3 + v_3 \otimes p_2$ are obtained from the adjoint representation and (as explained in Section 4) are not sufficient. Nevertheless proposition 5.4 predicts that if Y is perpendicular to the G -orbit on M , then

$$\rho(Y^\flat) = 0,$$

thanks to the definition of ρ (see Lemma 5.3); in that case

$$\Psi_* I_1 Y = -v_2 \otimes p_3 + v_3 \otimes p_2$$

which coincides with the irreducible representation of $\mathfrak{sp}(1)$ on $V = \mathbb{R}^3$.

6 Examples and applications

The apparent distinction between the points of view we have adopted in Section 4 and Section 5 disappears as soon as one compares (37) and (29). This

suggests that an intimate relationship exists between the two descriptions of the quaternionic structure: we are going to discuss now some examples which throw light on this link.

Let us consider the Wolf space

$$\mathbb{HP}^1 \cong \frac{Sp(2)}{Sp(1) \times Sp(1)} \cong \frac{SO(5)}{SO(4)} \cong S^4$$

and the action of the stabilizer $Sp(1) \times Sp(1)$ of a point N , with Lie algebra $\mathfrak{sp}(1)_+ \oplus \mathfrak{sp}(1)_- = \mathfrak{so}(4)$; this is a cohomogeneity 1 action, with generic orbits isomorphic to

$$S^3 \cong \frac{Sp(1) \times Sp(1)}{Sp(1)_\Delta}$$

where $Sp(1)_\Delta$ is the diagonal representation, and 2 singular orbits corresponding to a couple of antipodal points N, S . Let us choose at the point N any closed geodesic $\beta(t)$ connecting N to S : this will be orthogonal to any $Sp(1) \times Sp(1)$ orbit, and will intersect all of them (a *normal geodesic* in the language of [4], which in higher cohomogeneity is generalized by submanifolds called *sections*, see [12]). For instance, we can choose $N = e Sp(1) \times Sp(1)$, and take the geodesic corresponding to following copy of $U(1) \subset Sp(2)$:

$$g(t) = \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix} = \exp \begin{pmatrix} 0 & t & 0 & 0 \\ -t & 0 & 0 & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & -t & 0 \end{pmatrix}, \quad (39)$$

where the matrix on the right is denoted by tu . This subgroup generates a geodesic $\beta(t)$ connecting N ($t = 0$) with the south pole S ($t = \pi/2$) passing through the equator ($t = \pi/4$), and then backwards to N ($t = \pi$). The stabilizer of the $Sp(1) \times Sp(1)$ action is constant along $\beta(t)$ on points that are different from N and S , and coincides with $Sp(1)_\Delta$, both along $\beta(t)$ in \mathbb{HP}^1 and along $\mathfrak{u}(1)$ for the isotropy representation.

Let now e_i and f_i denote orthonormal bases of $\mathfrak{sp}(1)_+$ and $\mathfrak{sp}(1)_-$ respectively; as $\mathfrak{so}(4)$ is a subalgebra of $\mathfrak{sp}(2)$ corresponding to the longest root, the elements of the two copies of $\mathfrak{sp}(1)$ correspond to the following matrices:

$$e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad (40)$$

$$e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (41)$$

and

$$e_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}; \quad (42)$$

so if $e_i(t)$ and $f_i(t)$ denote an orthonormal basis of the isotropy subalgebra at $\beta(t)$ (given by $Ad_{g(t)}\mathfrak{so}(4)$), we get via the Killing metric:

$$\begin{aligned} \langle e_i, f_j(t) \rangle &= \delta_j^i \sin^2 t \\ \langle e_i, e_j(t) \rangle &= \delta_j^i \cos^2 t \\ \langle f_i, e_j(t) \rangle &= \delta_j^i \sin^2 t \\ \langle f_i, f_j(t) \rangle &= \delta_j^i \cos^2 t; \end{aligned}$$

in terms of Killing vector fields this implies

$$\pi_{S^2H}(\nabla \tilde{e}_i) = \sin^2 t f_i(t) \quad , \quad \pi_{S^2H}(\nabla \tilde{f}_i) = \cos^2 t f_i(t).$$

if we identify $S^2H \cong Ad_{g(t)}\mathfrak{sp}(1)_-$.

The conclusion is that along $\beta(t)$ the moment map for the action of $Sp(1) \times Sp(1)$ on $\mathbb{H}\mathbb{P}^1$ is given by

$$\mu(\beta(t)) = \sum_i \omega_i \otimes (\cos^2 t f_i + \sin^2 t e_i), \quad (43)$$

up to a constant. This is the only information that we need to reconstruct the moment map on the whole $\mathbb{H}\mathbb{P}^1$, as $\beta(t)$ intersects all the orbits and the moment map is equivariant.

We can now interpret these facts in terms of the induced map

$$\Psi : \mathbb{H}\mathbb{P}^1 \longrightarrow \mathbb{G}_3(\mathfrak{so}(4));$$

first of all we note that in this case $M_0 = M$, as the three vectors

$$B_i(t) = \cos^2 t f_i + \sin^2 t e_i \quad (44)$$

are linearly independent for all t ; moreover we observe that $\hat{\Phi}$ is a conformal mapping of bundles, as asked in the general hypotheses discussed in Section 3.

Recall from [23] that the critical manifolds for the gradient flow of the functional

$$\psi = \langle [v_1, v_2], v_3 \rangle$$

defined on $\mathbb{G}_3(\mathfrak{so}(4))$ are given by the maximal points $\mathfrak{sp}(1)_+$, $\mathfrak{sp}(1)_-$ and the submanifold

$$C_\Delta = \mathbb{R}\mathbb{P}^3 \cong \frac{Sp(1) \times Sp(1)}{\mathbb{Z}_2 \times Sp(1)_\Delta}$$

corresponding to the 3-dimensional subalgebra $\mathfrak{sp}(1)_\Delta$, for $\psi > 0$; the unstable manifold M_Δ emanating from this last one is 4-dimensional and isomorphic to

$$\frac{\mathbb{HP}^1 \setminus \{N, S\}}{\mathbb{Z}_2}.$$

A trajectory for the flow of $\nabla\psi$ is given by

$$V(x, y) = \text{span}\{xe_i + yf_i \mid x^2 + y^2 = 1, i = 1 \cdots 3\}, \quad (45)$$

therefore, comparing (45) with (44) we obtain that $\Psi(\mathbb{HP}^1) = M_\Delta \cup \mathfrak{sp}(1)_+ \cup \mathfrak{sp}(1)_-$; in particular:

$$\Psi(N) = \mathfrak{sp}(1)_- \quad (46)$$

$$\Psi(S) = \mathfrak{sp}(1)_+ \quad (47)$$

$$\Psi(\beta(\pi/4)) = \mathfrak{sp}(1)_\Delta. \quad (48)$$

Observation. The map Ψ is not injective. The points corresponding to t and $\pi - t$ are sent to the same 3-plane; so the principal orbits of type S^3 in \mathbb{HP}^1 are sent to the orbits of type \mathbb{RP}^3 in M_Δ . The map Ψ becomes injective on the orbifold $\mathbb{HP}^1/\mathbb{Z}_2$, nevertheless Φ_* is injective away from N, S .

Therefore the $Sp(1) \times Sp(1)$ orbit through $x_\Delta = \beta(\pi/4)$ is sent through Ψ to the critical orbit C_Δ ; we have

Proposition 6.1. *The differential*

$$T_{x_\Delta} \mathbb{HP}^1 \xrightarrow{\Psi_*} T_{\mathfrak{sp}(1)_\Delta} \mathbb{G}_3(\mathfrak{so}(4))$$

is a linear $Sp(1)_\Delta$ -invariant injective map. It coincides (up to a constant) with the map Q defined in (27), in terms of $Sp(1)_\Delta$ modules.

Proof. Let $\alpha(t)$ be any curve through x_Δ , then

$$g_* \cdot \Psi_* \alpha'(0) = \frac{d}{dt} g \cdot \Psi(\alpha(t)) = \frac{d}{dt} \Psi(g \cdot \alpha(t)) = \Psi_* g_* \cdot \alpha'(0)$$

for $g \in Sp(1)_\Delta \subset Sp(1) \times Sp(1)_{x_\Delta}$ where this last is the isotropy subgroup at x_Δ ; in this case the Lie algebra $\mathfrak{sp}(1)_\Delta$ of $Sp(1)_\Delta$, which is the stabilizer at $\beta(t)$ for any t , turns out to coincide with the image $\Psi(x_\Delta)$. The decomposition of the holonomy representation in terms of $Sp(1)_\Delta$ -modules is given in this case by

$$E \otimes H \cong \Sigma^1 \otimes \Sigma^1 \cong \Sigma^2 + \Sigma^0;$$

correspondingly, the decomposition of the Grassmannian's tangent space at $V = \mathfrak{sp}(1)_\Delta$ is given by

$$\begin{aligned} T_V \mathbb{G}_3(\mathfrak{so}(4)) &\cong V \otimes V^\perp \cong \mathfrak{sp}(1)_\Delta \otimes \Sigma^2 \cong \Sigma^2 \otimes \Sigma^2 \\ &\cong \Sigma^4 + \Sigma^2 + \Sigma^0, \end{aligned}$$

and Ψ_* sends injectively $\Sigma^2 + \Sigma^0$ in $\Sigma^4 + \Sigma^2 + \Sigma^0$; then, as a consequence of Schur's lemma, an isomorphism of $Sp(1)_\Delta$ -modules is unique up to a constant for each irreducible submodule, hence

$$\begin{cases} \Psi_* &= a Q & \text{on } \Sigma^2 \\ \Psi_* &= b Q & \text{on } \Sigma^0 \end{cases}$$

for some constants $a, b \in \mathbb{R}$. ■

An analogous situation holds for appropriate orbits in the following cases, which are all cohomogeneity 1 actions on classical Wolf spaces:

- $Sp(n)Sp(1)$ acting on $\mathbb{H}\mathbb{P}^n$;
- $Sp(n)$ acting on $\mathbb{G}_2(\mathbb{C}^{2n})$;
- $SO(n-1)$ acting on $\mathbb{G}_4(\mathbb{R}^n)$.

In the first case the orbit sent through Ψ to a critical submanifold of type C_Δ in the corresponding Grassmannian is one of the principal orbits S^{4n-1} , in the second and third case is one of the singular orbits, more precisely

$$\frac{Sp(n)}{Sp(n-2) \times U(2)} \quad \text{and} \quad \mathbb{G}_3(\mathbb{R}^{n-1}) \cong \frac{SO(n-1)}{SO(n-4) \times SO(3)}$$

respectively.

This situation can be generalized in the following sense: let G be a compact group acting by quaternionic isometries on a QK manifold M ; let G_x denote the stabilizer at the point $x \in M$; then $G_x \subset SO(T_x M)$ with respect to the quaternionic metric. Since QK manifolds are characterized by the condition $\text{Hol}(M)_x \subset Sp(n)Sp(1) \subset SO(T_x M)$, we have by hypothesis that $G_x \subset Sp(n)Sp(1)$. Now suppose that G_x contains some copy of $Sp(1)$ with nontrivial projection on the $Sp(n)$ factor. In the case that

$$\Psi(x) = \mathfrak{sp}(1)$$

and that a tubular neighborhood of G_x is sent to the unstable manifold (for $\psi > 0$) emanating from the critical manifold $C \subset \mathbb{G}_3(\mathfrak{g})$ corresponding to $\mathfrak{sp}(1)$, then we have a corresponding decomposition of $T_x M$ and $T_{\mathfrak{sp}(1)}\mathbb{G}_3(\mathfrak{g})$ in $Sp(1)$ -modules, and the differential Ψ_* coincides with Q up to determining $2k$ constants, 2 for each $Sp(1)$ -irreducible summand of the standard $Sp(n)$ module E .

Let us now decompose the holonomy representation in the case that $Sp(1)$ is the standard quaternionic subgroup, hence with trivial projection on the $Sp(n)$ factor. In this case E turns out to be a direct sum of trivial representations:

$$E \otimes H \cong (2n \Sigma^0) \otimes \Sigma^1 \cong 2n \Sigma^1$$

where $2\Sigma^1$ can be identified with the complexified algebra of Quaternions $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. Therefore going back to the real tangent bundle, we obtain the $Sp(1)$ invariant decomposition

$$T_x M \cong n \mathbb{H}. \quad (49)$$

The presence of the G -action allows to single out a quaternionic line of $T_x M$: this determines a quaternionic 1-dimensional distribution $\mathcal{N}_{\mathbb{H}}$ on M , or a section $\tau : M \longrightarrow \mathbb{H}\mathbb{P}(TM)$ of the associated $\mathbb{H}\mathbb{P}^{n-1}$ -bundle.

The distribution $\mathcal{N}_{\mathbb{H}}$ arises in the following way: recall that at a point $V \in \mathbb{G}_3(\mathfrak{g})$ with v_1, v_2, v_3 ON basis, we have

$$\text{grad } \psi = v_1 \otimes [v_2, v_3]^{\perp} + v_2 \otimes [v_3, v_1]^{\perp} + v_3 \otimes [v_1, v_2]^{\perp}.$$

Maintaining the general hypotheses considered in Sections 3 and 5, and assuming that Ψ_* is injective, let us define $X := \Psi_*^{-1}(\text{grad } \psi)$; then we have:

Lemma 6.2. *Suppose that $\Psi(x) = V$. Then the subspaces*

$$\begin{aligned} \text{span}\{\text{grad } \psi, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} &\subset T_V \mathbb{G}_3(\mathfrak{g}) \\ \text{span}\{X, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} &\subset T_x M \end{aligned}$$

are $Sp(1)$ invariant, hence quaternionic.

Proof. We need to prove that the endomorphisms of $S^2 H$ over x (or equivalently those of \mathbf{V} over V) preserve the respective subspaces; let us recall the description of I_1, I_2, I_3 given in Proposition 5.4, then

$$\begin{aligned} I_1(\text{grad } \psi) &= \frac{1}{\lambda} v_1 \otimes \rho((\text{grad } \psi)^{\flat}) - v_2 \otimes [v_1, v_2]^{\perp} + v_3 \otimes [v_3, v_1]^{\perp} \\ &= -v_2 \otimes [v_1, v_2]^{\perp} + v_3 \otimes [v_3, v_1]^{\perp} \\ &= -\tilde{v}_1, \end{aligned} \quad (50)$$

where the first summand vanishes thanks to the G -invariance of ψ , which implies that $\text{grad } \psi$ is orthogonal to the G orbits. Analogously, $I_2(\text{grad } \psi) = -\tilde{v}_2$ and $I_3(\text{grad } \psi) = -\tilde{v}_3$, and the quaternionic identities imply that the whole $\text{span}\{\text{grad } \psi, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ is preserved; the second inclusion follows from the injectivity and equivariance of Ψ . ■

In all the examples discussed above the distribution $\mathcal{N}_{\mathbb{H}}$ turns out to be integrable, with integral manifolds isomorphic to $\mathbb{H}\mathbb{P}^1$ embedded quaternionically in $\mathbb{H}\mathbb{P}^n$, $\mathbb{G}_2(\mathbb{C}^{2n})$ or $\mathbb{G}_4(\mathbb{R}^n)$ respectively.

For $Sp(1) \times Sp(1)$ acting on $\mathbb{H}\mathbb{P}^1$ the ditribution $\mathcal{N}_{\mathbb{H}}$ clearly coincides with the tangent bundle; in this case it is possible to describe the relationship between the two metrics and the $(\cdot)^{\sharp}$ endomorphism:

Proposition 6.3. *Let $M = \mathbb{H}\mathbb{P}^1 \setminus \{N, S\}$; consider the decomposition*

$$\begin{aligned} T_x M &\cong \text{span}\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} \oplus \text{span}\{X\} \\ &=: C_1 \oplus C_2 \end{aligned} \quad (51)$$

induced by the $Sp(1) \times Sp(1)$ action; then the map $\Psi : M \longrightarrow \mathbb{G}_3(\mathfrak{so}(4))$ satisfies the condition

$$\Psi^* \langle \cdot, \cdot \rangle_{\mathbb{G}_3} |_{C_i} = \eta_i(x) \langle \cdot, \cdot \rangle_M \quad i = 1, 2 \quad (52)$$

where $\eta_i(x)$ two real-valued $Sp(1) \times Sp(1)$ invariant function defined on M . The endomorphism (34) is just the multiplication by $\eta_i(x)$ on C_i .

Proof. The tangent space $T_V \mathbb{G}_3(\mathfrak{so}(4))$ along the unstable manifold can be seen as an irreducible $Sp(1)_\Delta$ -module, and Ψ_* as a morphism of $Sp(1)$ -modules. Schur's Lemma guarantees the uniqueness of an invariant bilinear form (up to a constant), for every irreducible submodule. Recall that

$$T_x M \cong \Sigma^2 \oplus \Sigma^0$$

as $Sp(1)_\Delta$ representations, corresponding to the splitting (51): therefore equation (52) holds, as both metrics are $Sp(1)_\Delta$ invariant. For the second assertion, let $Y \in C_i$:

$$\begin{aligned} (Y)^\sharp &= (\Psi^*((\Psi_* Y)^\flat))^\sharp \\ &= (\Psi^*(\langle \Psi_* Y, \cdot \rangle_{\mathbb{G}_3}))^\sharp \\ &= \eta_i(x) (\langle Y, \cdot \rangle_M)^\sharp \\ &= \eta_i(x) Y \end{aligned}$$

as required. ■

Observation. Equation (50) together with the equality $\|\text{grad } \psi\| = 3/2 \|\tilde{v}_i\|$ confirms that the endomorphisms I_i are *not* isometries for Grassmannian metric; hence $\Psi^* \langle \cdot, \cdot \rangle_{\mathbb{G}_3}$ and $\langle \cdot, \cdot \rangle_M$ can not coincide. Indeed,

$$\|\text{grad } \psi\|_{\mathbb{G}_3}^2 = \frac{3}{2} \|\tilde{v}_1\|_{\mathbb{G}_3}^2 = \frac{3}{2} \eta_2 \|\tilde{v}_1\|_M^2;$$

moreover

$$\|\text{grad } \psi\|_{\mathbb{G}_3}^2 = \eta_1 \|X\|_M^2$$

and $\|X\|_M = \|I_1 X\|_M = \|\tilde{v}_1\|_M$. Thus $\frac{\eta_1}{\eta_2} = \frac{3}{2}$. An analogous result is expected to hold in general.

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